Visualizing Monodromy

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Pomona Research in Mathematics Experience (PRiME)

Abstract

Arbitrarily choose loops γ around 0 and 1 in $\mathbb{P}^1(\mathbb{C})$ that start and end at x_0. Compute the paths that start at P_k, where P_k is the k^{th} point that corresponds to the inverse image of x_0. We refer to these paths as $\widetilde{\gamma}$. Monodromy describes the movement of $\widetilde{\gamma}$ and γ in correspondence to a Belyĭ map. A Belyĭ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a rational function with at most three critical values; we may assume these are $\{0,\,1,\,\infty\}$. The endpoints of our path correspond to a σ_0, σ_1 and $\sigma_\infty \in S_N$ such that $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$. This project sought to simplify the concept of monodromy for a general audience in the form of an eight minute video. Our movie not only provides visualizations of monodromy on the Riemann Sphere but highlights monodromy's connection to Belyĭ maps and Dessin d'Enfant through real world examples.

Background

- **Critical Values:** Consider a function $\beta : S \to \mathbb{P}^1(\mathbb{C})$ for the Riemann Sphere $S = \mathbb{P}^1(\mathbb{C})$. A critical point $P \in S$ satisfies $\beta'(P) = 0$. A critical value $w \in \mathbb{P}^1(\mathbb{C})$ is $w = \beta(P)$ the value of a critical point $P \in S$.
- **Belyĭ Maps:** Gennadiĭ Belyĭ [2] proved that a compact connected Riemann surface S of genus g is completely determined by the existence of a rational map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ which has three critical values. We say that a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ is a rational map with critical values $\{0,\,1,\,\infty\}$.
- **Dessin d'Enfant:** Following an idea from Alexander Grothendieck [3], we define a Dessin d'Enfant (French for "child's drawing") as a bipartite graph with "black" vertices $B = \beta^{-1}(0)$, "white" vertices $W = \beta^{-1}(1)$, midpoints of faces $F = \beta^{-1}(\infty)$, and edges $E = \beta^{-1}\big([0,1]\big)$. For our purposes, these Dessins d'Enfant are embedded on the sphere using stereographic projection.
- **Degree Sequences:** Choose $P \in B \cup W \cup F$. Denote the ramification index e_P as the number of edges at vertex P. A theorem of Adolf Hurwitz [4] asserts that, given a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ of degree N for a Riemann surface S of genus g,
$$N = \sum_{P\in B} e_P = \sum_{P\in W} e_P = \sum_{P\in F} e_P = |B| + |W| + |F| + (2\,g - 2).$$
The collection of the ramification indices can be collected into a multiset of multisets called the degree sequence $\mathcal{D}$.

Monodromy Triple

Sometimes a Degree Sequence does not correspond to a Belyĭ Map and sometimes a Degree Sequence corresponds to more than one Belyĭ Map. Adolf Hurwitz [4] proved that a Degree Sequence $\mathcal{D}$ corresponds to a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ of degree N on a Riemann Surface S of genus g if and only if there exist three permutations σ_0, σ_1, $\sigma_\infty \in S_N$ with the following properties:

i. The composition $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$ is the trivial permutation.

ii. The subgroup $\mathrm{Mon}(\beta) = \langle \sigma_0,\,\sigma_1,\,\sigma_\infty\rangle$ of the symmetric group S_N generated by them is a transitive subgroup. This is called the monodromy group of β.

iii. Each of these permutations is a product of disjoint cycles:
$$\begin{aligned}
\sigma_0 &= \prod_{P\in B} \left(b_{P,1}\, b_{P,2}\, \cdots\, b_{P,e_P}\right) & & B = \beta^{-1}(0)\\
\sigma_1 &= \prod_{P\in W} \left(w_{P,1}\, w_{P,2}\, \cdots\, w_{P,e_P}\right) & \text{where} \quad & W = \beta^{-1}(1)\\
\sigma_\infty &= \prod_{P\in F} \left(f_{P,1}\, f_{P,2}\, \cdots\, f_{P,e_P}\right) & & F = \beta^{-1}(\infty)
\end{aligned}$$

The tuple $(\sigma_0\,\sigma_1,\,\sigma_\infty)$ is called the monodromy triple associated with a Degree Sequence whenever it exists.

Dessin D'Enfant → Monodromy Triples

We now explain how to generate a Monodromy Triple $(\sigma_0\,\sigma_1,\,\sigma_\infty)$ from a Dessin D'Enfant. Let's use the following graph as an example:

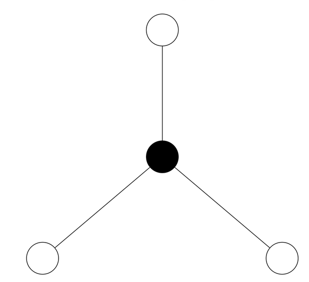


Figure 1: Dessin of **Degree** $N = 3$ & **Degree Sequence** $\mathcal{D} = \{\{3\},\,\{1,1,1\},\,\{3\}\}$ Corresponding to $\beta = z^3$

i. Label the edges of the graph from 1 to N.

ii. For each "black" vertex $P \in B$, read the labels counterclockwise to form the e_p-cycle $(b_{p,1}\, b_{p,2}\, ...b_{p,e_p})$.
$$\sigma_0 = \prod_{P\in B} \left(b_{P,1}\, b_{P,2}\, \cdots\, b_{P,e_P}\right)$$

iii. For each "white" vertex $P \in W$, read the labels counterclockwise to form the e_p-cycle $(w_{p,1}\, w_{p,2}\, ...w_{p,e_p})$.
$$\sigma_1 = \prod_{P\in W} \left(w_{P,1}\, w_{P,2}\, \cdots\, w_{P,e_P}\right)$$

iv. For each "face" $P \in F$, walk around the edge going clockwise by reading the labels which appear just after a "white" vertex to form the e_p-cycle $(f_{p,1}\, f_{p,2}\, ...f_{p,e_p})$. Combine all the faces to form the permutation
$$\sigma_\infty = \prod_{P\in F} \left(f_{P,1}\, f_{P,2}\, \cdots\, f_{P,e_P}\right)$$

v. As a check verify that $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$

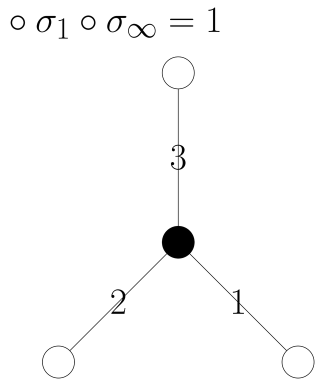


Figure 2: Permutation Triples: $\sigma_0 = (1\ 3\ 2)$, $\sigma_1 = (1)\,(2)\,(3)$, $\sigma_\infty = (1\ 2\ 3)$

Belyĭ Maps → Monodromy Triples

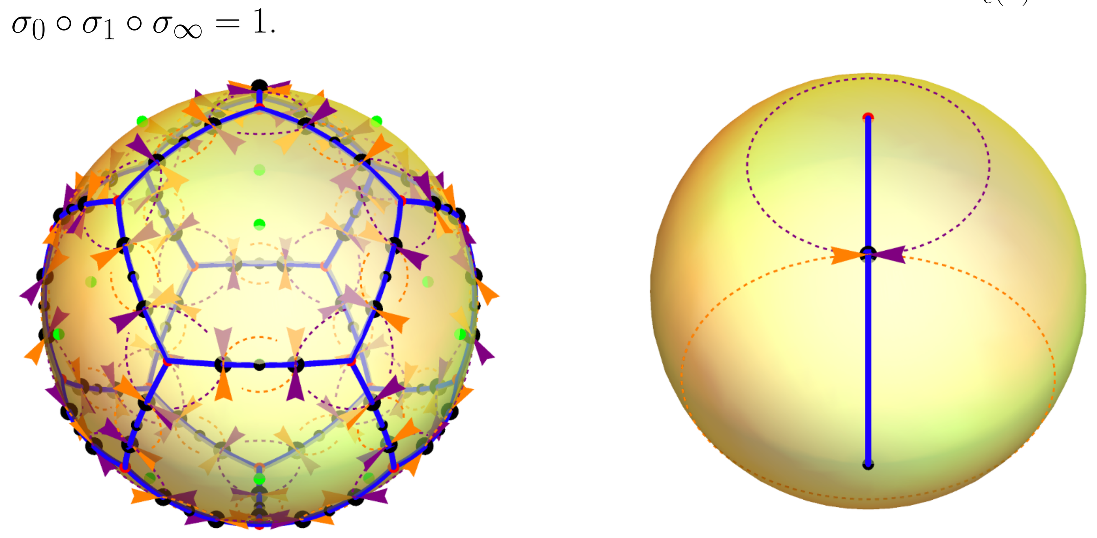
We can compute the Monodromy Triples $(\sigma_0,\,\sigma_1,\,\sigma_\infty)$ from a given Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ with the following steps:

#1. Choose $x_0 \neq 0,\,1,\,\infty$; and compute the inverse image $\beta^{-1}(x_0) = \{P_1,\,P_2,\,\ldots,\,P_N\}$.

#2. Choose loops γ around $\epsilon = 0,\,1$ in $\mathbb{P}^1(\mathbb{C})$ that start and end at x_0. For example, we often choose $\gamma_\epsilon(t) = \epsilon + (x_0 - \epsilon)\,e^{2\pi i t}$.

#3. For each P_k, compute those paths $\widetilde{\gamma}_\epsilon^{(k)}$ on the Riemann Surface S such that $\beta \circ \widetilde{\gamma}_\epsilon^{(k)} = \gamma_\epsilon$ and $\widetilde{\gamma}_\epsilon^{(k)}(0) = P_k$.

#4. Compute permutations σ_0, σ_1, $\sigma_\infty \in S_N$ satisfying $\widetilde{\gamma}_\epsilon^{(k)}(1) = P_{\sigma_\epsilon(k)}$ and $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$.



Motivating Question

How do you visualize Monodromy?

Monodromy in 2D

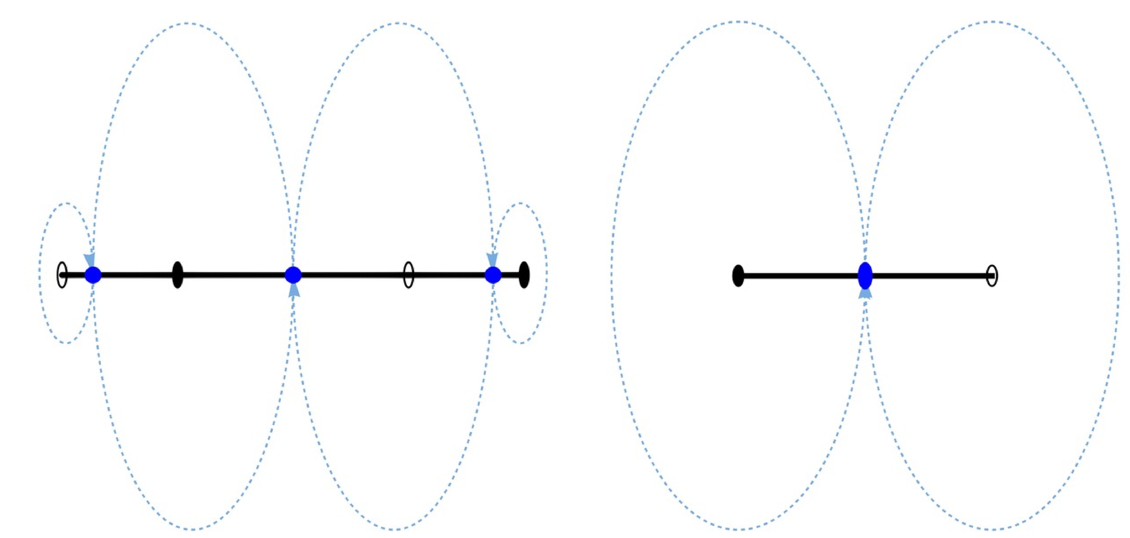


Figure 3: **Left:** Dessin of Degree $N = 3$ & Degree Sequence $\mathcal{D} = \{\{1,2\},\,\{1,2\},\,\{3\}\}$
Right: Dessin of Degree $N = 1$ & Degree Sequence $\mathcal{D} = \{\{1\},\,\{1\},\,\{1\}\}$

https://youtu.be/I5de7noo1T0

Monodromy in 3D

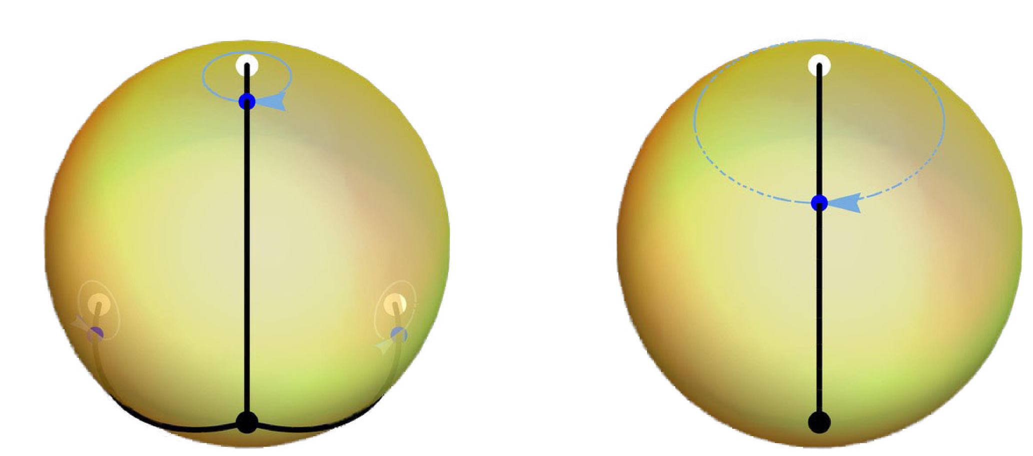
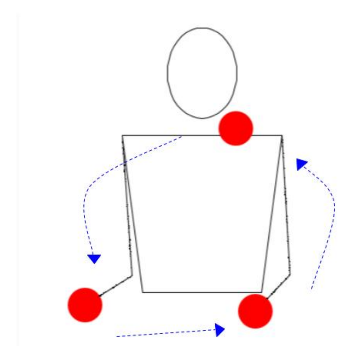


Figure 4: **Left:** Dessin of Degree $N = 3$ & Degree Sequence $\mathcal{D} = \{\{3\},\,\{1,1,1\},\,\{3\}\}$ stereographically projected on the sphere. **Right:** Dessin of Degree $N = 1$ & Degree Sequence $\mathcal{D} = \{\{1\},\,\{1\},\,\{1\}\}$ stereographically projected on the sphere.

https://youtu.be/hcHkVVDeslw

Monodromy in Life

Can you guess what Belyĭ map, Dessin D'Enfant and Monodromy triple this juggling motion corresponds to?



PRiME Time!

Take a look at the video we produced to explain the concept of Monodromy!

https://youtu.be/zUfb8AfGmPQ

Future Work

- Create a movie that contains visuals of monodromy on the torus
- Identify other juggling tricks as the monodromy of some Belyĭ map
- Compute on higher genera, degree, and other Riemann surfaces

References

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